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Exponential wave maps

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Abstract

We generalize wave maps to exponential wave maps. We compute the first and second variations of the exponential energy, and obtain results concerning the stability of exponential wave maps. We prove a theorem which relates wave maps, exponential wave maps, and the conservation law of second-order symmetric tensors. We show that if f is an exponential wave and a pseudo-weakly conformal map, then f is homothetic. We finally discuss the applications of exponential wave maps in relativity.

MSC: 58E20; 35K05; 58G11

Keywords: Exponential energy; Wave maps; Exponential wave maps

1. Introduction

Wave maps are harmonic maps on Minkowski spaces. We first studied Gu's paper [10] concerning harmonic maps on two-dimensional Minkowski spaces and became interested in wave maps. In this decade, there have been many new developments in wave maps achieved by Klainerman, Machedon [15–17], Shatah, Struwe [21,22], Tararu [25] Tao [23,24], Nahmod, Stefanov, and Uhlenbeck [18], etc. In this paper we generalize wave maps to exponential wave maps and explore the relationships between wave maps and exponential wave maps. We study exponential wave maps in the aspect of differential geometry instead of the analytic aspect of partial differential equations. Exponential wave maps are exponentially harmonic maps on Minkowski spaces. Exponentially harmonic maps were introduced by Eells and Lemaire [8] in 1990. These maps generalize usual harmonic maps given by Eells and Sampson [9] (Chiang's Ph.D. adviser) in 1964. In last four decades, there have been two reports on harmonic maps by Eells and Lemaire [6,7] in 1978 and 1988, and two books on harmonic maps, loop groups, and integrable systems by Guest [11] in 1997 and by Helein [12] in 2001. Chiang (with Andrew Ratto and Hongan Sun) studied harmonic maps and biharmonic maps in [1–5], and Yang (with Hong) studied exponentially harmonic maps in [13]. In 2002, Kanfon, Fuzfa, and Lambert [14] investigated exponentially harmonic maps, and constructed new models of exponentially harmonic maps which were coupled with gravitational fields with exponentially scalar fields.

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In Section 2 we compute the first and second variations of the exponential energy explicitly using tensor techniques which cannot be found in any other paper, and obtain the stability of exponentially harmonic maps in Theorems 2.3 and 2.4. We then review some results of exponentially harmonic maps. In Section 3 we give three examples of exponential wave maps. We obtain Propositions 3.1 and 3.2 concerning the stability of exponential wave maps by applying Theorems 2.3 and 2.4. We also prove Theorem 3.3 which relates wave maps, exponential wave maps, and the conservation law of second-order symmetric tensors. Afterwards, we prove that if f is an exponential wave map, then the associated energy–momentum tensor is conserved; cf. Theorem 3.5. We then use this theorem to prove Proposition 3.6, that if f is an exponential wave and pseudo-weakly conformal map, then f is homothetic. In Section 4 we discuss the applications of exponential wave maps in relativity in two cases: de Sitter spaces and Friedmann–Lemaitre spaces, by either approximating exponential wave maps by usual wave maps or by coupling them with gravitational fields with exponential scalar fields.

2. Exponentially harmonic maps

An exponentially harmonic map $f : M \to N$ from a m-dimensional Riemannian manifold (M^m, g_{ij}) to a n-dimensional Riemannian manifold $(N^n, h_{\alpha\beta})$ is a critical point of the exponential energy functional

$$E(f) = \int_M e^{|\mathrm{d}f|^2}\mathrm{d}v = \int_M e^{h_{\alpha\beta} f_i^\alpha f_j^\beta g^{ij}}\mathrm{d}v, \tag{2.1}$$

where $\mathrm{d}v$ is the volume element on M. In order to compute the Euler–Lagrange equation, we consider a one-parameter family of maps $f_t \in C^\infty(M \times I, N)$ such that f_t is the endpoint of a segment starting at $f(x)$ determined in length and direction by the vector field $\dot f(x)$, and such that the compact support of $\dot f(x)$ is contained in the interior of M. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(f_t)|_{t=0} &= 2\int_M e^{|\mathrm{d}f_t|^2}(\mathrm{d}f_t, \nabla_t \mathrm{d}f_t)|_{t=0} = 2\int_M e^{|\mathrm{d}f|^2}(\mathrm{d}f, \nabla\dot f)\mathrm{d}v\\ &= 2\left(\int_M \mathrm{div}(w)\mathrm{d}v - \int_M e^{|\mathrm{d}f|^2}((\tau f, \dot f) + (\nabla|\mathrm{d}f|^2, \mathrm{d}f), \dot f)\right)\mathrm{d}v\\ &= -2\int_M e^{|\mathrm{d}f|^2}((\tau f + (\nabla|\mathrm{d}f|^2, \mathrm{d}f), \dot f))\mathrm{d}v = 0, \quad \forall \dot f \end{aligned}\tag{2.2}$$

by the divergence theorem, which implies that $\tau f + (\nabla|\mathrm{d}f|^2, \mathrm{d}f) = 0$, where $\tau^\alpha(f) = g^{ij} f^\alpha_{i|j} = g^{ij}((f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k) + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j)$ is the tension field, ∇ is the connection on $T^*(M) \otimes f^{-1}TN$ induced by the Levi-Civita connections on M and N, $\mathrm{div}(w) = w^j_{|j}$, $w^j = e^{h_{\alpha\beta} f_i^\alpha f_j^\beta g^{ij}} h_{\alpha\beta} f_i^\alpha \dot f^\beta g^{ij}$ is a vector field on M.

Definition 2.1. A map $f : M \to N$ between two Riemannian manifolds is *exponentially harmonic* if it satisfies

$$\tau f + (\nabla|\mathrm{d}f|^2, \mathrm{d}f) = 0, \tag{2.3a}$$

i.e., in terms of local coordinates it satisfies

$$\begin{aligned} &g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right) + g^{il}g^{jm}h_{\beta\gamma}\frac{\partial f^\alpha}{\partial x^l}\frac{\partial f^\gamma}{\partial x^m}\frac{\partial^2 f^\beta}{\partial x^i \partial x^j}\\ &\quad - g^{il}g^{jm}h_{\beta\gamma}\Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^l}\frac{\partial f^\beta}{\partial x^m}\frac{\partial f^\gamma}{\partial x^k} + g^{ij}g^{lm}h_{\beta\gamma}\Gamma'^\beta_{\mu\nu}\frac{\partial f^\mu}{\partial x^i}\frac{\partial f^\nu}{\partial x^l}\frac{\partial f^\gamma}{\partial x^m}\frac{\partial f^\alpha}{\partial x^j} = 0, \end{aligned}\tag{2.3b}$$

where Γ^k_{ij} and $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connections on M and N respectively.

We first note that if $|\mathrm{d}f|^2$ is constant, then f is exponentially harmonic iff it is harmonic by (2.3). Some properties of exponentially harmonic maps are different from those of usual harmonic maps. When $\dim(M) = m = 2$, if we perform a conformal shift on the metric $g \mapsto \rho g$, then both energy $\int_M |\mathrm{d}f|^2\mathrm{d}v$ and the harmonic map are conformal invariant. But, for an exponentially harmonic map, the energy (2.1) changes completely.

Example 1. If $u : R^2 \to R$ is an exponentially harmonic function, (2.3) leads to

$$(1+u_x^2)u_{xx} + 2u_x u_y u_{xy} + (1+u_y^2)u_{yy} = 0. \tag{2.4}$$

By the method of separable variables, the solutions are in the form $u(x, y) = F(x) + G(y)$. It follows from (2.4) that $(1 + (F_x)^2)F_{xx} = -(1 + (G_y)^2)G_{yy} = \lambda$. Let $p = F_x, q = G_y$ and substitute these into the equation. By straightforward computation, we can get

$$F(x) = \frac{1}{4\lambda}\{[H_+(x;\lambda;c_1) + H_-(x:\lambda;c_1)]^4 + 2[H_+(x;\lambda;c_1) + H_-(x;\lambda;c_1)]^2\} - k_1,$$

where $H_\pm(x;\lambda;c_1) = \{\frac{3}{2}(c_1+\lambda x) \pm (1 + \frac{9}{4}(c_1+\lambda x)^2)^{1/2}\}^{1/3}$. Similarly, we have

$$G(y) = -\frac{1}{4\lambda}\{[H_+(y:-\lambda;c_2) + H_-(y;-\lambda;c_2)]^4 + 2[H_+(y;-\lambda;c_2) + H_-(y;-\lambda;c_2)]^2\} - k_2.$$

We also have $p = H_+(x;\lambda;c_1) + H_-(x;\lambda;c_1), q = H_+(y;-\lambda;c_2) + H_-(y:-\lambda;c_2)$. Therefore, $u(x, y)$ can be written with a parametric representation:

$$x = \frac{1}{\lambda}\left(\frac{p^3}{3} + p - c_1\right), \qquad y = \frac{-1}{\lambda}\left(\frac{q^3}{3} + q - c_2\right)$$

$$u(x, y) = \frac{1}{4\lambda}(p^4 + 2p^2 - q^4 - 2q^2) + \text{constant}.$$

It is easy to check that $u(x, y)$ is not harmonic.

Assume that $f = f_0$ is exponentially harmonic and that $\xi = \frac{\partial f}{\partial t}$ has compact support contained in the interior of M. The components of $\nabla_t \tau f$ are $f^\alpha_{i|j|t} = \frac{\partial f^\alpha_{i|j}}{\partial t} + \Gamma'^\alpha_{\mu\gamma} f^\mu_{i|j}\xi^\gamma$. We use the curvature formula on $M \times I \to N$ and have $f^\alpha_{i|j|t} = f^\alpha_{i|t|j} + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \xi^\mu$. But $f^\alpha_{i|t} = f^\alpha_{t|i} = \xi^\alpha_{|i}$; therefore the trace of $\nabla_t \tau f$ has components $g^{ij}\xi^\alpha_{|i|j} + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j g^{ij}\xi^\mu$. Denote the first term by $(\triangle\xi)^\alpha$. Then we can compute the second variation of the energy from (2.2):

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} &= -\int_M \frac{\mathrm{d}}{\mathrm{d}t}(e^{|\mathrm{d}f_t|^2}(\tau f_t + (\nabla|\mathrm{d}f_t|^2, \mathrm{d}f_t), \xi))|_{t=0}\mathrm{d}v \\ &= -\int_M (e^{\mathrm{d}f|^2}(\nabla_t(\tau f + (\nabla|\mathrm{d}f_t|^2, \mathrm{d}f_t), \xi)|_{t=0} + (\tau f + (\nabla|\mathrm{d}f|^2, \mathrm{d}f), \nabla_t\xi)) \\ &\quad + e^{|\mathrm{d}f|^2} 2(\nabla_t \mathrm{d}f, \mathrm{d}f)(\tau f + (\nabla|\mathrm{d}f|^2, \mathrm{d}f), \xi))\mathrm{d}v.\end{aligned} \tag{2.5}$$

Since f is exponentially harmonic at $t = 0$, the second and third terms of (2.5) disappear and by substituting the components of $\nabla_t \tau f$ we have

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} = -\int_M e^{|\mathrm{d}f|^2}((\triangle\xi + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j g^{ij}\xi^\mu, \xi) + (\nabla(\nabla|\dot f|^2, \dot f), \xi))\mathrm{d}v \tag{2.6}$$

$$= \int_M e^{|\mathrm{d}f|^2}((\nabla\xi, \nabla\xi) - R'_{\alpha\beta\gamma\mu}\xi^\alpha f^\beta_i f^\gamma_j \xi^\mu + 2(\nabla\xi, \xi)^2)\mathrm{d}v, \tag{2.7}$$

from integration by parts, $\mathrm{d}(\nabla\xi, \xi) = (\triangle\xi, \xi) + (\nabla\xi, \nabla\xi)$ for the first term, and $(\nabla(\nabla|\dot f|^2, \dot f), \xi) = (\nabla((2\nabla\dot f, \dot f), \dot f), \xi) = (\nabla((2\nabla\xi, \xi), \xi), \xi) = 2(\triangle\xi, \xi^3) + 4(((\nabla\xi, \nabla\xi), \xi), \xi)$, and integration by parts, $\mathrm{d}(\nabla\xi, \xi^3) = (\triangle\xi, \xi^3) + 3(\nabla\xi, \xi^2\nabla\xi)$ for the second term. We can rewrite (2.6) as

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} = \int_M e^{|\mathrm{d}f|^2}(-(J_f(\xi), \xi) + 2(\nabla\xi, \xi)^2)\mathrm{d}v, \tag{2.8}$$

where

$$J_f(\xi) = \triangle\xi + R'(\mathrm{d}f, \mathrm{d}f)\xi = g^{ij}\xi^\alpha_{|i|j} + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \xi^\mu g^{ij}, \tag{2.9}$$

which is a linear equation for ξ. Solutions of $J_f(\xi) = 0$ are called Jacobi fields.

Definition 2.2. Let $f : M \to N$ be an exponentially harmonic map. If $\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(f_t)|_{t=0} \geq 0$, then f is stable.

Theorem 2.3. *Let $f : M \to N$ be an exponentially harmonic map. If N has non-positive sectional curvature (i.e. $R'_{\alpha\beta\gamma\mu}\lambda^\alpha\eta^\beta\lambda^\gamma\eta^\mu \leq 0$ for arbitrary λ, η), then f is stable.*

Proof. It follows from (2.7).

Theorem 2.4. *Let $f : M \to N$ be an exponentially harmonic map. If $\xi = \dot{f}$ is a Jacobi field, then f is stable.*

Proof. It follows from (2.8).

Proposition 2.5 ([13]). *Let $f : M \to N$ be an exponentially harmonic map, where M is compact without boundary, $Ricc^M \geq 0$, and $Riem^N \leq 0$.*

(1) *Then f is totally geodesic.*
(2) *If $Ricc^M$ is positive at at least one point of M, then f is constant.*
(3) *If $Riem^N$ is everywhere negative, then f is either constant or maps M onto a closed geodesic of N.*

Proposition 2.6 ([13]). *Let $M^m \subset R^{m+1}$ be a hypersurface which has m principal curvatures with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, satisfying $\lambda_m < \lambda_1 + \cdots + \lambda_{m-1}$. If $f : N \to M$ is a stable exponentially harmonic map with $|\mathrm{d}f|^2 < \frac{1}{2x_m^2} \min_{1\leq i\leq m}\{\lambda_i(\sum_{j=1}^m \lambda_j - 2\lambda_i)\}$, then f is constant.*

Proposition 2.7 (*Liouville* [13]). *Let $f : R^m \to N$ be an exponentially harmonic map. If f has finite energy and $|\mathrm{d}f|^2 \leq \frac{m}{2} - 1$, then f is constant.*

3. Exponential wave maps

Let $R^{m,1}$ be a $m+1$-dimensional Minkowski space with the metric $g_{ij} = (-1, 1, 1, \ldots, 1)$ and the coordinates $x^0 = t, x^1, x^2, \ldots x^m$, and $(N, h_{\alpha\beta})$ be an n-dimensional Riemannian manifold. A wave map is a harmonic map on the Minkowski space $R^{m,1}$ with the energy

$$E(f) = \int_{R^{m,1}} h_{\alpha\beta}(-f_t^\alpha f_t^\beta + f_{x^i}^\alpha f_{x^i}^\beta)\mathrm{d}t\mathrm{d}x. \tag{3.1}$$

The Euler–Lagrange equation describing the critical point of (3.1) is

$$\tau_\Box^\alpha(f) = -\Box f^\alpha + \Gamma'^\alpha_{\beta\gamma}(-f_t^\beta f_t^\gamma + f_{x^i}^\beta f_{x^i}^\gamma) = 0 \tag{3.2}$$

which is the wave map equation, where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x^i}^2}$ is the d'Alembertian. The wave map equation is invariant with respect to the dimensionless scaling $f(t, x) \to f(ct, cx)$, $c \in R$. However, the energy is scale invariant only in dimension $m = 2$. If $f : R^{m,1} \to N$ is a wave map, by (2.9) $\xi = \frac{\partial f}{\partial s}$ is a Jacobi field on the Minkowski space $R^{m,1}$ satisfying

$$J_f^\alpha(\xi) = -\Box\xi^\alpha + R'^\alpha_{\beta\gamma\mu}(-f_t^\beta f_t^\gamma \xi^\mu + f_i^\beta f_i^\gamma \xi^\mu) = 0,$$

where $\{f_s\} : R^{m,1} \times I \to N$ is a one-parameter family of maps.

An exponential wave map $f : R^{m,1} \to N$ is an exponentially harmonic map on the Minkowski space $R^{m,1}$ with the exponential energy from (2.2):

$$E(f) = \int_{R^{m,1}} e^{h_{\alpha\beta}(-\frac{\partial f^\alpha}{\partial t}\frac{\partial f^\beta}{\partial t} + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^i})}\mathrm{d}t\mathrm{d}x. \tag{3.3}$$

The Euler–Lagrange equation describing the critical point of (3.3) from (2.3) is

$$\tau_\Box(f) + \langle\nabla(-|\partial_t f|_h^2 + |\partial_{x^i} f|_h^2), \mathrm{d}f\rangle = 0, \tag{3.4a}$$

i.e., in terms of local coordinates it satisfies

$$-\frac{\partial^2 f^\alpha}{\partial t^2}+\sum_{i=1}^m\frac{\partial^2 f^\alpha}{\partial {x^i}^2}+\Gamma'^\alpha_{\beta\gamma}\left(-\frac{\partial f^\beta}{\partial t}\frac{\partial f^\gamma}{\partial t}+\sum_{i=1}^m\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^i}\right)$$
$$+\sum_{i=0}^m\sum_{j=0}^m g^{ii}g^{jj}h_{\beta\gamma}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\frac{\partial^2 f^\beta}{\partial x^i\partial x^j}+\sum_{i=0}^m\sum_{j=0}^m g^{ii}g^{jj}h_{\beta\gamma}\Gamma'^\beta_{\mu\nu}\frac{\partial f^\mu}{\partial x^i}\frac{\partial f^\nu}{\partial x^j}\frac{\partial f^\gamma}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}=0. \tag{3.4b}$$

Example 2. If the energy density $e(f)=-f_t^\alpha f_t^\beta+f_i^\alpha f_i^\beta$ is constant, then $\tau_\Box(f)=0$ if and only if $\tau_\Box(f)+(\nabla(-|\partial_t f|_h^2+|\partial_{x^i}f|_h^2),\mathrm{d}f)=0$. Therefore, f is a wave map with constant energy if and only if it is an exponential wave map.

Example 3. If $u:R^{1,1}\to R$ is an exponential wave function, (3.4) becomes

$$(1+u_x^2)u_{xx}-2u_tu_xu_{tx}-(1-u_t^2)u_{tt}=0. \tag{3.5}$$

By the method of separable variables, the solutions are $u(t,x)=F(t)+G(x)$. We have from (3.5) that $(1+F_t^2)F_{tt}=(1-G_x^2)G_{xx}=\lambda'$. By computation similar to that of Example 1, $u(t,x)$ can be written with a parametric representation:

$$t=\frac{1}{\lambda'}\left(\frac{p^3}{3}+p-c_3\right),\qquad x=\frac{1}{\lambda'}\left(-\frac{q^3}{3}+q-c_4\right)$$
$$u(t,x)=\frac{1}{4\lambda'}(p^4+2p^2-q^4+2q^2)+\text{constant}.$$

It is easy to check that $u(x,y)$ is not a wave function.

Example 4. Let $M=R^{1,1}$ and N be a surface of revolution in three-dimensional Euclidean space with the metric

$$\mathrm{d}s^2=[1+(\mathrm{d}h/\mathrm{d}z)^2]\mathrm{d}z^2+h^2(z)\mathrm{d}\phi^2,$$

where $r=h(z)$ is the equation of N in cylindrical coordinates. We can extend the example of a wave map given by Gu in [10] to an exponential wave map. The first equation of (3.4b) becomes

$$-\frac{\partial^2 z}{\partial t^2}+\frac{\partial^2 z}{\partial x^2}+\frac{h'h''}{1+h'^2}\left[-\left(\frac{\partial z}{\partial t}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2\right]-\frac{hh'}{1+h'^2}\left[-\left(\frac{\partial \phi}{\partial t}\right)^2+\left(\frac{\partial \phi}{\partial x}\right)^2\right]$$
$$+(1+h'^2)\left[\left(\frac{\partial z}{\partial t}\right)^2\frac{\partial^2 z}{\partial t^2}-\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t\partial x}-\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x\partial t}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x^2}\right]$$
$$+h^2\left[\frac{\partial z}{\partial t}\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}-\frac{\partial z}{\partial t}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial t\partial x}-\frac{\partial z}{\partial x}\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t\partial x}+\frac{\partial z}{\partial x}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2}\right]$$
$$+(1+h'^2)\left[\frac{h'h''}{1+h'^2}\left(\left(\frac{\partial z}{\partial t}\right)^4-\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}+\left(\frac{\partial z}{\partial x}\right)^4\right)\right.$$
$$\left.-\frac{hh'}{1+h'^2}\left(\left(\frac{\partial \phi}{\partial t}\right)^2\left(\frac{\partial z}{\partial t}\right)^2-\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}-\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}+\left(\frac{\partial \phi}{\partial x}\right)^2\left(\frac{\partial z}{\partial x}\right)^2\right)\right]$$
$$+h^2\left[\frac{h'}{h}\left(\left(\frac{\partial z}{\partial t}\right)^2\left(\frac{\partial \phi}{\partial t}\right)^2-\frac{\partial z}{\partial t}\left(\frac{\partial \phi}{\partial x}\right)\frac{\partial \phi}{\partial t}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}\frac{\partial \phi}{\partial t}\frac{\partial z}{\partial t}\frac{\partial \phi}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2\left(\frac{\partial \phi}{\partial x}\right)^2\right)\right.$$
$$\left.-\frac{h'}{h}\left(\frac{\partial \phi}{\partial t}\right)^2\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial \phi}{\partial t}\right)^2\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial \phi}{\partial t}\right)^2\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial x}\right)^2\left(\frac{\partial z}{\partial x}\right)^2\right]=0. \tag{3.6}$$

Consider the special initial conditions: when $t = 0$,

$$\phi = x, \quad \frac{\partial \phi}{\partial t} = 0, \quad z = k, \quad \frac{\partial z}{\partial t} = \alpha, \tag{3.7}$$

where k, α are constants. The solution is invariant with respect to rotation around the z-axis, and therefore $\phi = x$, $z = z(t)$. (3.6) for $z(t)$ has the form

$$-\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} - \frac{h'h''}{1+h'^2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - \frac{hh'}{1+h'^2} + (1+h'^2)\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + h'h''\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^4 = 0, \tag{3.8}$$

and the initial conditions are $z(0) = k$, $(\frac{\mathrm{d}z}{\mathrm{d}t})_0 = \alpha$. (3.8) admits a first integral

$$\begin{aligned}
&-(1+h'^2)\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - h^2 + (1+h'^2)^2 \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^4 \\
&\quad = \left(\frac{1}{\sqrt{2}}(1+h'^2)\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - \frac{1+\sqrt{1+2h^2}}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}(1+h'^2)\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - \frac{1-\sqrt{1+2h^2}}{\sqrt{2}}\right) \\
&\quad = \left(\frac{1}{\sqrt{2}}(1+h'^2(k))\alpha^2 - \frac{1+\sqrt{1+2h^2(k)}}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}(1+h'^2(k))\alpha^2 - \frac{1-\sqrt{1+2h^2(k)}}{\sqrt{2}}\right).
\end{aligned} \tag{3.9}$$

The solutions can be represented as

$$\int_k^z \frac{(1+h'^2)^{1/2}\mathrm{d}z}{\sqrt{(1+h'^2(k))\alpha^2 - \sqrt{1+2h^2(k)} + \sqrt{1+2h^2(z)}}} = t \tag{3.10}$$

or

$$\int_k^z \frac{(1+h'^2)^{1/2}\mathrm{d}z}{\sqrt{(1+h'^2(k))\alpha^2 + \sqrt{1+2h^2(k)} - \sqrt{1+2h^2(z)}}} = t. \tag{3.11}$$

If $\alpha^2(1+h'^2(k)) - \sqrt{1+2h^2(k)} + \sqrt{1+2h^2(z)} > 0$ in (3.10) for all z, then all of N can be covered. Otherwise, the surface is covered partially. Similarly for (3.11). The second equation of (3.4) yields to $0 = 0$ under the special initial conditions in (3.7).

Let $f : R^{m,1} \to N$ be an exponential wave map. f is *stable* if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} \geq 0$.

Proposition 3.1. *If $f : R^{m,1} \to N$ is an exponential wave map such that $R'_{\alpha\beta\gamma\mu}\xi^\alpha(f_i^\beta f_i^\gamma - f_t^\beta f_t^\gamma)\xi^\mu \leq 0$, then f is stable.*

Proof. By (2.6) and (2.7) we have

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} = \int_{R^{m,1}} e^{h_{\alpha\beta}(f_i^\alpha f_i^\beta - f_t^\alpha f_t^\beta)}((\nabla\xi, \nabla\xi) - R'_{\alpha\beta\gamma\mu}\xi^\alpha(f_i^\beta f_i^\gamma - f_t^\beta f_t^\gamma)\xi^\mu + 2(\nabla\xi, \xi)^2)\mathrm{d}t\mathrm{d}x$$

and the result follows from the hypotheses.

Proposition 3.2. *Let $f : R^{m,1} \to N$ be an exponential wave map. If $\xi = \dot{f}$ is a Jacobi field on the Minkowski space $R^{m,1}$, then f is stable.*

Proof. By (2.8) we have

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} = \int_{R^{m,1}} e^{h_{\alpha\beta}(f_i^\alpha f_i^\beta - f_t^\alpha f_t^\beta)}(-(J_f(\xi), \xi) + 2(\nabla\xi, \xi)^2)\mathrm{d}t\mathrm{d}x,$$

where

$$J_f^\alpha(\xi) = -\Box\xi^\alpha + R'^{\alpha}_{\beta\gamma\mu}(-f_t^\beta f_t^\gamma \xi^\mu + f_i^\beta f_i^\gamma \xi^\mu).$$

If $J_f(\xi) = 0$, then $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(f_t)|_{t=0} \geq 0$.

We next study the relationships among wave maps, exponential wave maps, and the conservation law of second-order symmetric tensors as follows:

Theorem 3.3. *Let $f : R^{m,1} \to N$ be a non-degenerate map (i.e. $\mathrm{d}f \neq 0$). Then any two conditions of the following imply the third:*

(1) *f is a wave map.*
(2) *f is an exponential wave map.*
(3) *The second-order symmetric tensor $S_f = |\mathrm{d}f|^2(f^*h - \frac{1}{4}|\mathrm{d}f|^2 g)$ is conserved, i.e., $\mathrm{div}(S_f) = 0$, where $g = (-1, 1, \dots, 1)$, $|\mathrm{d}f|^2 = -|\frac{\partial f}{\partial t}|^2 + |\frac{\partial f}{\partial x^i}|^2$.*

Proof. I. (1) and (2) $\Rightarrow$ (3): Let $x^0 = t, x^1, x^2, \dots, x^m$ be the coordinates in $R^{m,1}$, and $e_0 = \frac{\partial}{\partial t}, e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots e_m = (0, 0, \dots, 1)$. Set

$$S = |\mathrm{d}f|^2 \left(f^*h - \frac{1}{4}|\mathrm{d}f|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \right),$$

where I is an $m \times m$ identity matrix. For $X \in T(R^{m,1}) = R^{m,1}$, we compute

$$\begin{aligned}
(\mathrm{div}\, S_f)(X) &= (\nabla_{e_i} S)(e_i, X) = \nabla_{e_i} |\mathrm{d}f|^2 \left(f^*h - \frac{1}{4}|\mathrm{d}f|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \right)(e_i, X) \\
&= \nabla_{e_i} |\mathrm{d}f|^2 \left((f_* e_i, f_* X) - \frac{1}{4}|\mathrm{d}f|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \right)(e_i, X) + |\mathrm{d}f|^2 \big((f_* \nabla_{e_i} e_i, f_* X) \\
&\quad + (f_* e_i, f_* \nabla_{e_i} X) - \frac{1}{4}\nabla_{e_i} |\mathrm{d}f|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}(e_i, X) \Big) (\tau_{\square}(f) = (\nabla \mathrm{d}f)(e_i, e_i) = (\nabla_{e_i} \mathrm{d}f)(e_i)) \\
&= ((\nabla |\mathrm{d}f|^2, \mathrm{d}f) + |\mathrm{d}f|^2 \tau_{\square}(f), f_* X) + |\mathrm{d}f|^2 (f_* e_i, (\nabla_{e_i} \mathrm{d}f) X) \\
&\quad - \frac{1}{2}\nabla_X \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x^i}\right|^2 \right)\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x^i}\right|^2 \right) \\
&= ((\nabla |\mathrm{d}f|^2, \mathrm{d}f) + |\mathrm{d}f|^2 \tau_{\square}(f), f_* X) + \frac{1}{2}(\nabla_X |\mathrm{d}f|^2|)|\mathrm{d}f|^2 \\
&\quad - \frac{1}{2}\nabla_X \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2 \right)\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x^i}\right|^2 \right) \\
&= \left(\left(\nabla \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2 \right), \mathrm{d}f \right) + \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2 \right) \tau_{\square}(f), f_* X \right),
\end{aligned} \tag{3.12}$$

where the second and third terms are cancelled out. Therefore,

$$\mathrm{div}\, S_f = \left(\left(\nabla \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2 \right), \mathrm{d}f \right) + \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2 \right) \tau_{\square}(f), \mathrm{d}f \right)$$

Thus, (1) and (2) imply (3).

II. (2) and (3) $\Rightarrow$ (1): If f is an exponential wave map and S is conserved, then

$$\tau_{\square}(f) + \left(\nabla \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \frac{\partial f}{\partial x}\right|^2 \right), \mathrm{d}f \right) = 0 \tag{3.13}$$

$$\left(\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \frac{\partial f}{\partial x}\right|^2 \right) \tau_{\square}(f) + \left(\nabla \left(-\left|\frac{\partial f}{\partial t}\right|^2 + \frac{\partial f}{\partial x}\right|^2 \right), \mathrm{d}f \right), \mathrm{d}f \right) = 0. \tag{3.14}$$

Because f is non-degenerate (i.e. $\mathrm{d}f \neq 0$), so

$$\left(\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2\right)\tau_\Box(f) + \left(\nabla\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2\right), \mathrm{d}f\right)\right) = 0,$$

and thus $(\nabla(-|\frac{\partial f}{\partial t}|^2 + |\frac{\partial f}{\partial x}|^2), \mathrm{d}f) = -(-|\frac{\partial f}{\partial t}|^2 + |\frac{\partial f}{\partial x}|^2)\tau_\Box(f) = -|\mathrm{d}f|^2\tau_\Box(f)$. We substitute it into (3.13), and get $(1 - |\mathrm{d}f|^2)\tau_\Box(f) = 0$. Suppose that f is not a wave map; there exists a point $p \in R^{m,1}$ such that $\tau_\Box(f) \neq 0$ by the continuity of $\tau_\Box(f)$. Therefore, there exists a neighborhood U of p such that $|\mathrm{d}f|_U^2 = 1$, but (3.4) implies $\tau_\Box(f)|_U = 0$; a contradiction!

III. (1) and (3) $\Rightarrow$ (2) is obvious. $\Box$

Definition 3.4. $f : R^{m,1} \to N$ is *pseudo-weakly conformal* if there is a smooth function $\mu : R^{m,1} \to R$ such that $f^*h = \mu \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. f is *homothetic* if μ is constant.

If $f : R^{m,1} \to N$ is *pseudo-weakly conformal*, then we get

$$S_f = \frac{1}{4}(m-1)(5-m)\mu^2 g, \quad g = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}. \tag{3.15}$$

We have: (1) $S_f = 0$ if and only if $m = 1$ or 5 and f is pseudo-weakly conformal. (2) If $f : R^{m,1} \to N$ is pseudo-weakly conformal such that $m \neq 1, 5$ and S_f is conserved, then f is homothetic. By Theorem 3.3(3) div $S_f = 0$ and (3.15) we find

$$0 = \frac{1}{2}(m-1)(5-m)\mu\mu_{,j}g_{ij} = 0 \quad (0 \leq i \leq m),$$

whence $\mathrm{d}\mu = 0$ on $R^{m,1}$. Therefore, μ is constant.

The energy–momentum tensor associated with $f : R^{m,1} \to N$ is defined by $T(f) = e^{|\mathrm{d}f|^2}(g - 2f^*h)$, where $g = (-1, 1, 1, \ldots 1)$, $|\mathrm{d}f|^2 = -|\frac{\partial f}{\partial t}|^2 + |\frac{\partial f}{\partial x^i}|^2$. We have

Theorem 3.5. *If $f : R^{m,1} \to N$ is an exponential wave map, then $T(f)$ is conserved.*

Proof. Let $x^0 = t, x^1, x^2, \ldots, x^m$ be the coordinates in $R^{m,1}$, and $e_0 = \frac{\partial}{\partial t}$, $e_1 = (1, 0, \ldots, 0), \ldots e_m = (0, 0, \ldots 0, 1)$. Set

$$T(f) = e^{|\mathrm{d}f|^2}\left(\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} - 2f^*h\right). \tag{3.16}$$

For $X \in R^{m,1}$ we compute

$$\begin{aligned}
\operatorname{div} T(f)(X) &= \nabla_{e_i} T(f)(e_i, X) = \nabla_{e_i}\left[e^{|\mathrm{d}f|^2}\left(\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} - 2f^*h\right)(e_i, X)\right] \\
&= \nabla_{e_i} e^{|\mathrm{d}f|^2}\left(\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}(e_i, X) - 2(f_*e_i, f_*X)\right) - 2e^{|\mathrm{d}f|^2}(\nabla_{e_i} f^*h)(e_i, X) \\
&= e^{|\mathrm{d}f|^2}\left[\nabla|\mathrm{d}f|^2\left(\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}(e_i, X) - 2\nabla_{e_i}|\mathrm{d}f|^2(f_*e_i, f_*X)\right) - 2\nabla_{e_i}(f_*e_i, f_*X)\right] \\
&= e^{|\mathrm{d}f|^2}\left[2\left(\left(\nabla\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) + \left(\nabla\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i}\right)\right) - 2\nabla_{e_i}|\mathrm{d}f|^2(f_*e_i, f_*X)\right. \\
&\quad \left. -\ 2(\nabla_{e_i} f_*e_i, f_*X) - 2(f_*e_i, \nabla_{e_i} f_*X)\right] \\
&= 2e^{|\mathrm{d}f|^2}[((\nabla_X \mathrm{d}f)e_i, f_*e_i) - (\nabla|\mathrm{d}f|^2, \mathrm{d}f), f_*X - (f_*e_i, \nabla_{e_i} f_*X) - (\tau_\Box(f), f_*X)] \\
&= -2e^{-|\frac{\partial f}{\partial t}|^2 + |\frac{\partial f}{\partial x}|^2}\left(\tau_\Box(f) + \left(\nabla\left(-\left|\frac{\partial f}{\partial t}\right|^2 + \left|\frac{\partial f}{\partial x}\right|^2\right), \mathrm{d}f\right), f_*X\right),
\end{aligned}$$

where the first and third terms are cancelled out and $\tau_{\Box}(f) = \nabla_{e_i} f_* e_i$. Hence, if f is an exponential wave map, then $T(f)$ is conserved, i.e. $\operatorname{div} T(f) = 0$.

Proposition 3.6. *If $f : R^{m,1} \to N$ is an exponential wave and a pseudo-weakly conformal map such that $\mu \neq \frac{1}{2} - \frac{1}{m-1} (m \neq 1)$, then f is homothetic.*

Proof. By (3.16) $T(f) = e^{(m-1)\mu} g(1-2\mu)$, $g = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$ due to the pseudo-weak conformality of f. Since f is an exponential wave, then by Theorem 3.5 $\operatorname{div} T(f) = 0$ we have

$$e^{(m-1)\mu}(m-1)\mu_{,j} g_{ij}(1-2\mu) + e^{(m-1)\mu} g_{ij}(-2\mu_{,j}) = e^{(m-1)\mu} \mu_{,j} g_{ij}((m-1)(1-2\mu) - 2) = 0.$$

If $\mu \neq \frac{1}{2} - \frac{1}{m-1} (m \neq 1)$, then $\mu_{,j} g_{ij} = 0\, (0 \leq i \leq m)$ which implies that $\mathrm{d}\mu = 0$, and hence μ is constant.

The proof of the Liouville-type theorem for an exponentially harmonic map in Proposition 2.7 depends on the assumption that f has finite energy, i.e., $\int_{R^m} |\mathrm{d}f|^2 \mathrm{d}v < \infty$, which implies $|\mathrm{d}f|^2 = 0$, and therefore, f is constant. If we apply it to an exponential wave map $f : R^{m,1} \to N$ under the assumption, $\int_{R^{m,1}} (-|f_t|^2 + \sum_{i=1}^m |f_{x_i}|^2) \mathrm{d}t \mathrm{d}x < \infty$, which implies that $\sum_{i=1}^m |f_{x_i}|^2 = |f_t|^2$. Then f is not necessarily constant.

4. The applications of exponential wave maps

Let $f : R^{m,1} \to (N, h_{\alpha\beta})$ be a C^∞ map between two Riemannian manifolds. If we want to relate our context with physics, we need to modify the exponential energy (3.3) as follows:

$$\begin{aligned} E'_\lambda(f) &= \int_{R^{m,1}} e^{\lambda h_{\alpha\beta}(-\frac{\partial f^\alpha}{\partial t}\frac{\partial f^\beta}{\partial t} + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^i})} \mathrm{d}t\mathrm{d}x \approx \lambda \int_{R^{m,1}} \Big[h_{\alpha\beta}(-f_t^\alpha f_t^\beta + f_i^\alpha f_i^\beta) \\ &\quad + \frac{\lambda}{2}(h_{\alpha\beta}(-f_t^\alpha f_t^\beta + f_i^\alpha f_i^\beta))^2 + \frac{\lambda^2}{6}(h_{\alpha\beta}(-f_t^\alpha f_t^\beta + f_i^\alpha f_i^\beta))^3 + \cdots \Big] \mathrm{d}t\mathrm{d}x. \end{aligned} \tag{4.1}$$

When λ is small enough, the Euler–Lagrange equations for $E'_\lambda(f)$ lead to equations which approximate those of usual wave maps. The equations derived from E'_λ can be obtained from (3.4) via the shift $f \mapsto \sqrt{\lambda} f (\lambda > 0)$.

General relativistic solutions can be locally embedded in Ricci-flat five-dimensional spaces. This is important in establishing local generality for the work recently developed by Wesson [19], whereby vacuum $(4+1)$-dimensional field equations give rise to $(3+1)$-dimensional equations with sources. We briefly describe the mathematical structure of Wesson's schemes [20] using the following two postulates.

Postulate 4.1. The fundamental space in which an ordinary four-dimensional spacetime is locally and isometrically embedded may be described with a five-dimensional manifold M_5. The line element of this space is given by $\mathrm{d}\tilde{s}^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$ which can be put, at least locally, in the form $\mathrm{d}\tilde{s}^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + \epsilon\phi^2\mathrm{d}\psi^2$, where $\{a, b\}$ and $\{i, j\}$ run from 0 to 4, and 0 to 3 respectively, $x^a = (x^i, \psi)$ are coordinates, $g_{ij} = g_{ij}(x^i)$, $\phi = \phi(x^a)$, $\epsilon^2 = 1$.

Postulate 4.2. The fundamental five-dimensional space satisfies the vacuum field equations ${}^{(5)}\tilde{R}_{ab} = 0$.

Theorem 4.3 (*Campbell*). *Any analytic n-dimensional Riemannian space can be locally embedded in an $(n+1)$-dimensional Ricci-flat space (cf. [20]).*

We would like to discuss the applications of exponential wave maps in relativity as follows:

Case 1: Let S_4 be a four-dimensional de Sitter spacetime with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2\sqrt{\Lambda/3}t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{4.2}$$

where Λ is the cosmological constant. We consider an exponential wave map $f : S_4 \to R$ (globally) approximated by a usual wave map which is the extremal of the functional E'_λ in (4.1). It satisfies a modified version of wave map equation (3.4) via the shift $f \mapsto \sqrt{\lambda} f (\lambda > 0)$, which is written as

$$\ddot{f}(1 + \lambda\dot{f}^2) + 6\sqrt{\Lambda/3}\dot{f} = 0, \quad (\dot{f} = f_t) \tag{4.3}$$

if the map is restricted to $f = f(t)$. This yields $\ln(\dot{f}) + \frac{\lambda}{2}\dot{f}^2 = -6\sqrt{\Lambda/3}\,t + c_1$, and thus, $\dot{f}e^{\frac{\lambda}{2}\dot{f}^2} = c_2 e^{-6\sqrt{\Lambda/3}\,t}$.

(i) When $t \to \infty$, $f(t) \to$ const.
(ii) When $t \to 0$ and λ is small, $f(t) \approx c_2 t + c_3$ which is regular at $t = 0$.

By Postulates 4.1, 4.2 and Theorem 4.3, S_4 can be embedded in a five-dimensional Ricci-flat space S_5 [19][p. 333] and [20, p. 374] with the metric

$$\mathrm{d}\tilde{s}^2 = \Lambda\psi^2/3\mathrm{d}t^2 - \Lambda\psi^2/3e^{2\sqrt{\Lambda/3}t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) - \mathrm{d}\psi^2 \tag{4.4}$$

which induces the metric (4.2) on the hypersurface $\psi = \psi_0 = \pm\sqrt{3/\Lambda}$, $\psi^2 = 3/\Lambda$.

Case 2: (1) Let M_4 be a Friedmann–Lemaitre space with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right), \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\gamma^2. \tag{4.5}$$

It is known that an exponentially harmonic map $f : M_4 \to R$ (globally, $k = 0$) is not regular at $t = 0$. Kanfon, Fuzfa and Lambert [14] considered this exponentially harmonic map $f : M_4 \to R$ on the F–L space without matter coupled with an exponentially scalar field which can make f regular at $t = 0$.

(2) (a) The F–L space M_4 can be locally embedded in five-dimensional space M_5 by Postulates 4.1, 4.2, and Theorem 4.3 with the metric

$${}^{(5)}\mathrm{d}\tilde{s}^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right) + \epsilon\phi^2\mathrm{d}\psi^2, \quad \epsilon^2 = 1. \tag{4.6}$$

In particular, if $k = 0$, $a^2(t) = t$, $\phi^2(t) = 1/t$, the M_5 has the metric by [20, p. 372]

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}t^2 - t(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{\epsilon}{t}\mathrm{d}\psi^2.$$

We assume that M_5 has the metric ($a(t)$ is a function of t and $\phi(t) = 1/a(t)$)

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{\epsilon}{a^2(t)}\mathrm{d}\psi^2. \tag{4.7}$$

Take $\epsilon = -1$ (space-like). We consider an exponential wave map $f : M_5 \to R$ (locally) approximated by a usual wave map which is the extremal of the functional E'_λ in (4.1). It satisfies the following modified version of the wave map equation via the shift $f \mapsto \sqrt{\lambda}f\,(\lambda > 0)$:

$$\ddot{f}(1 + \lambda\dot{f}^2) + 4\frac{\dot{a}}{a}\dot{f} = 0$$

if the map is restricted to $f = f(t)$. This gives $a^4(t) = \frac{c_4}{|\dot{f}|}e^{-\frac{\lambda}{2}\dot{f}^2}$. For instance, take $a(t) = a_0(t/t_0)^{1/2}$, and $t_0 = \frac{1}{2H_0}$ where H_0 is the present Hubble constant ($\frac{\dot{a}}{a}|_{t=t_0} = H_0$). Then we have

$$|\dot{f}|e^{\frac{\lambda}{2}\dot{f}^2} = \frac{1}{dt^2}, \quad d = \frac{a_0^4}{c_4 t_0^2}.$$

(i) When $t \to \infty$, $f(t) \to$ const.
(ii) When $t \to 0$, and λ is small, $f(t) \approx f_0 \pm \frac{1}{\mathrm{d}t}$, which is not regular at $t = 0$.

(b) If we consider $f : M_5 \to R$ coupled with an exponentially scalar field using the metric (4.6),

$$S(f) = -\frac{1}{2\kappa}\int\sqrt{-g}\mathrm{d}^4x\mathrm{d}y\left\{\left(\tilde{R} - \exp\left(\frac{\lambda}{2}\partial_a f\partial^b f\right) - \Lambda\right) + \mathcal{L}_{\mathrm{mat}}\right\} \tag{4.8}$$

where $y = \psi$ represents the fifth new coordinate and the integration restricts to the hypersurface Σ_4 defined by $\psi = \psi_0 =$ constant, κ is a coupling constant, Λ is a modified cosmological constant: $\Lambda = 2\kappa(2\Lambda_0 - 1)$ with Λ_0 is the usual cosmological constant, $\mathcal{L}_{\mathrm{mat}}$ is the Lagrangian density for matter. By [19, Section 3] (4.8) reduces to

$$S(f) = -\frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x \left\{\left(R\phi - \exp\left(\frac{\lambda}{2}\partial_i f \partial^j f\right) - \Lambda\right) + \mathcal{L}_{\mathrm{mat}}\right\}, \phi = 1/a(t). \tag{4.9}$$

The variation of $S(f)$ leads to Einstein's equations:

$$\left(R_{ij} - \frac{1}{2}Rg_{ij}\right)\phi = \frac{1}{2}g_{ij}\left\{\left(-e^{\frac{\lambda}{2}\partial_i f\partial^j f} - \Lambda\right) + \lambda\partial_i f\partial_j f e^{\frac{\lambda}{2}\partial_i f\partial^j f}\right\} - R\delta\phi + \kappa T_{ij}^{(\mathrm{mat})} \tag{4.10}$$

where $T_{ij}^{(\mathrm{mat})}$ is the energy–momentum tensor for the matter. Let us assume that $f = f(t)$. Then the field equations can be written as

$$\frac{1}{a}3\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a}\left(\frac{3k}{a^2}\right) = \rho - \frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2}(1-\lambda\dot{f}^2) - \frac{\Lambda}{2} - 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)(-a^{-2})\dot{a} \tag{4.11}$$

$$\frac{1}{a}\left(\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right) + \frac{1}{a}\left(\frac{k}{a^2}\right) = -p - \frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2} - \frac{\Lambda}{2} \tag{4.12}$$

$$\ddot{f}(1+\lambda\dot{f}^2) + 4\frac{\dot{a}}{a}\dot{f} = 0, \tag{4.13}$$

where ρ is the mass–energy density of matter, and p is the pressure of the fluid.

In particular, if $k = 0$ and the F–L space is without matter, the above field equations become

$$\frac{1}{a}3\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2}(1-\lambda\dot{f}^2) - \frac{\Lambda}{2} - 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)(-a^{-2})\dot{a} \tag{4.14}$$

$$\frac{1}{a}\left(\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right) = -\frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2} - \frac{\Lambda}{2}. \tag{4.15}$$

Let $y = \dot{f}$ and let $H = \frac{\dot{a}}{a}$ be the Hubble constant. We have $\dot{H} + H^2 = \frac{\ddot{a}}{a}$. Then we can rewrite (4.14), (4.15) and (4.13) as

$$3H^2 = \left(-\frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2}(1-\lambda\dot{f}^2) - \frac{\Lambda}{2}\right)a + 6(\dot{H} + H^2)H + 6H^3 \tag{4.16}$$

$$a = \frac{H^2 + 2(\dot{H} + H^2)}{-\frac{1}{2}e^{\frac{\lambda}{2}\dot{f}^2} - \frac{\Lambda}{2}} \tag{4.17}$$

$$H = \frac{-\dot{y}(1+\lambda y^2)}{4y}. \tag{4.18}$$

Substitute (4.17) and (4.18) into (4.16) and, as λ is very small, it becomes

$$\frac{1}{2}\frac{\ddot{y}}{y} + \frac{3}{8}\frac{\ddot{y}\dot{y}}{y^2} + \frac{9}{16}\left(\frac{\dot{y}}{y}\right)^3 - \frac{1}{2}\left(\frac{\dot{y}}{y}\right) \approx 0. \tag{4.19}$$

Let $z = \frac{\dot{y}}{y}$, and we have $\dot{z} + z^2 = \frac{\ddot{y}}{y}$. We can rewrite (4.19)

$$\frac{1}{2}\dot{z} + \frac{3}{8}\dot{z}z + \frac{15}{16}z^3 \approx 0$$

which can be put as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{-15z^3}{8+6z}.$$

By integrating we get $(t+c)z^2 - \frac{6}{15}z - \frac{4}{15} = 0$. Substituting $z = \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}$ and solving for $\frac{\mathrm{d}y}{\mathrm{d}t}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{16 \pm \sqrt{256 + 240(t+c)}}{30(t+c)} y.$$

This gives solutions

$$y = c_5 e^{\frac{16 \pm \sqrt{256+240(t+c)}}{30(t+c)}}, \quad a(t) = c_6 e^{Ht}.$$

When $t = 0$, $f'(0) = c_5 e^{\frac{16 \pm \sqrt{256+240c}}{30c}}$ exists if $c \neq 0$. Therefore, $f(t)$ is differentiable and regular at $t = 0$. It is interesting to note that the coupling of f with the gravitational field can make f regular at $t = 0$, which is not the case for the uncoupled situation.

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